Complex Analysis: Resit Exam

Aletta Jacobshal 03, Friday 8 April 2016, 14:00 – 17:00
Exam duration: 3 hours

Instructions — read carefully before starting

- Do not forget to very clearly write your **full name** and **student number** on each answer sheet and on the envelope. Do not seal the ennvelope.
- The exam consists of 6 questions; answer all of them.
- The total number of points is 100 and 10 points are "free". The exam grade is the total number of points divided by 10.
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explain why the conditions for using such results are satisfied.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (12 points)

(a) (6 points) Verify that the function $f(z) = (z+i)^2$ satisfies the Cauchy-Riemann equations.

(b) (6 points) Compute the Taylor series of the function $f(z) = (z+i)^2$ around $z_0 = 1 \in \mathbb{C}$. What is the domain where this Taylor series converges?

Question 2 (18 points)

Consider the function

$$f(z) = \frac{e^{-iz}}{z^2 + 4}.$$

(a) (6 points) Compute the residue of $f(z)$ at each one of the singularities of $f(z)$.

(b) (12 points) Compute the principal value

$$\text{pv} \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2 + 4}\, dx.$$

Question 3 (14 points)

Consider the branch $f(z) = \mathcal{L}_{2\pi}(z)$ of the logarithm.

(a) (6 points) Compute $f(-e)$ and $f'(-e)$. Write the results in Cartesian form.

(b) (8 points) Evaluate the limits $\lim_{\varepsilon \to 0^+} f(x + i\varepsilon)$ and $\lim_{\varepsilon \to 0^+} f(x - i\varepsilon)$ for $x > 0$.

Question 4 (14 points)

Consider the function

$$f(z) = \frac{z^2}{z - 2}.$$

(a) (4 points) Determine the singularities of $f(z)$ and their type.

(b) (10 points) Compute the Laurent series $\sum_{j=-\infty}^{\infty} a_j z^j$ of the function $f(z)$ in the domain $|z| > 2$. What is the value of a_{-1}?

Question 5 (16 points)

(a) (6 points) Given the function

$$f(z) = \frac{z^3\,(z-3i)\,(z+1)^2}{z^2+2i},$$

evaluate the integral

$$\int_C \frac{f'(z)}{f(z)}\,dz,$$

where C is the positively oriented circular contour with $|z| = 2$.

(b) (10 points) Use Rouché's theorem to show that the polynomial $P(z) = z^3 + \varepsilon(z^2+1)$, where $0 < \varepsilon < 8/5$, has exactly 3 roots in the disk $|z| < 2$.

Question 6 (16 points)

(a) (8 points) Show that

$$\left|\int_C \frac{e^z}{\bar{z}+2}\,dz\right| \leq \pi e^2,$$

where C is the positively oriented circle $|z-1| = 1$.

(b) (8 points) Suppose that $f(z)$ is an entire function such that $f(z)/z^2$ is bounded for $|z| \geq R$, where $R > 0$. Show that $f(z)$ is a polynomial of degree at most 2.

End of the exam (Total: 90 points)